A Direct Projection Method for the Block Kaczmarz Algorithm

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This paper proposes a new implementation of the block Kaczmarz algorithm for solving systems of linear equations by the least squares method. Each iteration of the proposed algorithm can be considered as the solution of a sub-system defined by a specific arrowhead matrix. This sub-system is solved in an effective way using the direct projection method.

1 Main Results

For a matrix $A \in \mathbb{R}^{m\times n}$ and $f \in \mathbb{R}^m$ let $Au = f$ be a overdetermined consistent system of linear equation, where $u \in \mathbb{R}^n$ and $m \geq n$. Let's consider S as a positive-definite matrix (hence, invertible), whose Cholesky decomposition $S = LL^T$, where $S \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{n\times n}$. Consider matrix L as a right preconditioner, then we can write $Au = f$ as $AL\tilde{u} = f$ where $\tilde{u} = L^{-1}u$. To solve this system, we use the well-known Kaczmarz algorithm in a block modification. It's interesting that each iteration of this algorithm we can consider as

$$\begin{pmatrix} I_n & L^T A_{j(k)}^T \\ A_{j(k)}L & \left(1-\alpha_k^{-1}\right) A_{j(k)} S A_{j(k)}^T \end{pmatrix} \begin{pmatrix} \tilde{u}_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} \tilde{u}_k \\ f_{j(k)} \end{pmatrix} \Leftrightarrow \; Q_k x^k = b^k, \quad (1)$$

where

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_p \end{pmatrix},\; f = \begin{pmatrix} f_1 \\ \vdots \\ f_p \end{pmatrix},\; A_i = \begin{pmatrix} a^T_{(i-1)\cdot l+1} \\ \vdots \\ a^T_{(i-1)\cdot l+l} \end{pmatrix},\; f_i = \begin{pmatrix} f_{(i-1)\cdot l+1} \\ \vdots \\ f_{(i-1)\cdot l+l} \end{pmatrix}$$

and $A_i \in \mathbb{R}^{l\times n}$, $f_i \in \mathbb{R}^l$, p is the number of blocks, l is the number of rows in the block A_i, $m = l \cdot p$, $i = 1, 2, \ldots, p$, and $j(k) : \mathbb{N}_0 \to \{1, 2, \ldots p\}$ is a surjection from a set of natural numbers with zero to a set of block indexes. We will assume that $\alpha_k \in (0, 2)$ for the convergence of iterations.

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Theorem 1. *The iterations (1) are equivalent to the iterations of block Kaczmarz algorithm with a relaxation parameter α_k and can be written as:*

$$\begin{aligned} l < n: \quad \tilde{u}_{k+1} &= \tilde{u}_k - \alpha_k L^+_{j(k)} \left(L_{j(k)} \tilde{u}_k - f_{j(k)} \right), \\ l \geq n: \quad u_{k+1} &= u_k - \alpha_k A^+_{j(k)} \left(A_{j(k)} u_k - f_{j(k)} \right), \end{aligned}$$

where $L_{j(k)} = A_{j(k)} L$ and $\alpha_k \in (0, 2)$, the $[\cdot]^+$ denotes the Moore-Penrose pseudoinverse, and $k = 0, 1, \ldots, \infty$. In general, the proof is obvious but an especially interesting in the final step for $l \geq n$. We have to recall the famous results from [1] here, and we should note that $LL^+_{j(k)} = L \left(A_{j(k)} L \right)^+ = LL^+ \left(A_{j(k)} LL^+ \right)^+ = A^+_{j(k)}$.

Theorem 2. *The linear system $Q_k x^k = b^k$ is nonsingular and for any k* $\det(Q_k) = \left(-\alpha_k^{-1} \right)^l \det \left(A_{j(k)} S A^T_{j(k)} \right)$. It's follow from Aitken block-diagonalization formula for Q_k and exploiting the fact that the determinant of the triangle matrix block is the product of the determinants of its diagonal blocks.

For solving linear system (1) at each iteration it is proposed to use the direct projection method [3, 2]. It is worth to note that the proposed matrix equation (1) has some interesting properties and can be solved using the well-known algorithms [4], some of them can be effective enough.

If the initial approximation is fulfilled for a especial matching conditions, then the first n iterations of the direct projection method are redundant. Moreover, we can assume that if $l \geq n$, then some of preconditioning techniques don't appear to be effective in any case.

References

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